

THE COMPLEXITY OF ISOMORPHISM BETWEEN COUNTABLY BASED PROFINITE GROUPS

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ABSTRACT. A topological group G is profinite if it is compact and totally disconnected. Equivalently, G is the inverse limit of a surjective system of finite groups carrying the discrete topology. We discuss how to represent a countably based profinite group as a point in a Polish space. Then we study the complexity of isomorphism using the theory of Borel reducibility in descriptive set theory. For topologically finitely generated profinite groups this complexity is the same as the one of identity for reals. In general, it is the same as the complexity of isomorphism for countable graphs.

1. INTRODUCTION

A compact topological group G is called profinite if the clopen sets form a basis for the topology. Equivalently, the open normal subgroups form a base of nbhds of the identity. Since open subgroups of a compact group have finite index, this means that G is the inverse limit of its system of finite quotients with the natural projection maps.

All profinite groups in this paper will be countably based without further mention. This makes it possible to represent them as points in a Polish space. If X, Y are Polish spaces and E, F equivalence relations on X, Y respectively, one writes $(X, E) \leq_B (Y, F)$ (or simply $E \leq_B F$) if there is a Borel function $g: X \rightarrow Y$ such that $uEv \leftrightarrow guFgv$ for each $u, v \in X$.

The goal of this paper is to study the complexity of (topological) isomorphism for various classes of profinite groups.

For topologically finitely generated profinite groups, the complexity of isomorphism turns out to be rather low down: it is the same as the one of identity for reals. For the upper bound we rely on a result of Lubotzky [9] to show that isomorphism is the orbit equivalence relation of the action of a compact group, hence closed, and hence Borel below identity on \mathbb{R} . To show hardness we use another result of Lubotzky [10].

The lowness of isomorphism for finitely generated profinite groups contrasts with the complexity of isomorphism for abstract finitely generated groups, which is as hard as possible: by a result of Thomas and Velickovic [14] it is Borel complete for Borel equivalence relations with all classes countable.

S_∞ denotes the Polish group of permutations of ω . An S_∞ orbit equivalence relation is an orbit equivalence relation of a continuous action S_∞ on a Polish space. We show that isomorphism of general profinite groups is complete for S_∞ orbit equivalence relations. So, in a certain sense, the profinite groups, being either finite or of size the continuum, behave like

countable structures; for instance, isomorphism of countable graphs is well known to be S_∞ complete.

2. PRELIMINARIES

In a group that is finitely generated as a profinite group, all subgroups of finite index are open [12]. This deep theorem implies that the topological structure is determined by the group theoretic structure. In particular, all abstract homomorphisms between such groups are continuous.

2.1. Completion. We follow [13, Section 3.2]. Let G be a group, \mathcal{V} a set of normal subgroups of finite index in G such that $U, V \in \mathcal{V}$ implies that there is $W \in \mathcal{V}$ with $W \subseteq U \cap V$. We can turn G into a topological group by declaring \mathcal{V} a basis of neighbourhoods (nbhds) of the identity. In other words, $M \subseteq G$ is open if for each $x \in M$ there is $U \in \mathcal{V}$ such that $xU \subseteq M$.

The completion of G with respect to \mathcal{V} is the inverse limit

$$G_{\mathcal{V}} = \varprojlim_{U \in \mathcal{V}} G/U,$$

where \mathcal{V} is ordered under inclusion and the inverse system is equipped with the natural maps: for $U \subseteq V$, the map $p_{U,V}: G/U \rightarrow G/V$ is given by $gU \mapsto gV$. The direct limit can be seen as a closed subgroup of the direct product $\prod_{U \in \mathcal{V}} G/U$ (where each group G/U carries the discrete topology), consisting of the functions α such that $p_{U,V}(\alpha(gU)) = gV$ for each g . Note that the map $g \mapsto (gU)_{U \in \mathcal{V}}$ is a continuous homomorphism $G \rightarrow G_{\mathcal{V}}$ with dense image; it is injective iff $\bigcap \mathcal{V} = \{1\}$.

If the set \mathcal{V} is understood from the context, we will usually write \widehat{G} instead of $G_{\mathcal{V}}$.

2.2. The Polish space of profinite groups.

Definition 2.1. Let \widehat{F}_k be the free profinite group in k generators x_0, \dots, x_{k-1} ($k < \omega$).

Thus, \widehat{F}_k is the profinite completion of the abstract free group on k generators. Any topologically finitely generated profinite group can be written in the form

$$\widehat{F}_k/R$$

for some k and a closed normal subgroup R of \widehat{F}_k .

Definition 2.2. Let \widehat{F}_ω be the free profinite group on a sequence of generators x_0, x_1, x_2, \dots converging to 1 [13, Thm. 3.3.16].

Thus, \widehat{F}_ω is the completion in the sense of Subsection 2.1 of the free group F_ω on generators x_0, x_1, \dots with respect to the system of normal subgroups of finite index that contain almost all the x_i . Any profinite group G has a generating sequence $\langle g_i \rangle_{i \in \mathbb{N}}$ converging to 1. This is easy to see using coset representatives for a descending sequence of open normal subgroups that form a fundamental system of nbhds of 1_G , as in the proof of Thm. 6.1 below. (Also see [13, Prop. 2.4.4 and 2.6.1].) By the universal property of the completion, the map from the abstract free group induced by $x_i \rightarrow g_i$

extends to a continuous epimorphism $\widehat{F}_\omega \rightarrow G$. So G can be written in the form

$$\widehat{F}_k/R$$

where R is a closed normal subgroup of \widehat{F}_k .

In the following let $k \leq \omega$. For a compact Polish group G (such as \widehat{F}_k) let $\mathcal{N}(G)$ denote the collection of normal closed subgroups of G . It is standard to equip $\mathcal{N}(G)$ with the structure of a Polish space, in fact of a compact countably based space, as we describe next.

The compact subsets of a complete separable metric space (M, δ) form a complete metric space $\mathcal{K}(M)$ with the usual Hausdorff distance

$$\delta_H(A, B) = \max(\sup_{x \in A} \inf_{y \in B} \delta(x, y), \sup_{y \in B} \inf_{x \in A} \delta(x, y)).$$

Let $D \subseteq M$ be countable dense. Then $\mathcal{K}(M)$ contains as a dense subset the set of finite subsets of D . Since $\mathcal{K}(M)$ is a metric space, this implies that $\mathcal{K}(M)$ is countably based. If M is compact then $\mathcal{K}(M)$ is compact as well.

It is well-known that every compact Polish group G has a compatible bi-invariant metric δ ; that is, δ induces the given topology, and $\delta(xg, yx) = \delta(gx, gy) = \delta(x, y)$ for any $x, y, g \in G$.

We verify that $\mathcal{N}(G)$ is closed in $\mathcal{K}(G)$, using that the metric on G is bi-invariant. Firstly, if a sequence of subgroups $\langle U_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}(G)$ converges $U \in \mathcal{K}(G)$, then U is a subgroup. For suppose $a, b \in U$. For each $\epsilon > 0$, for sufficiently large n we can choose $a_n, b_n \in U_n$ with $d(a_n, a) < \epsilon$ and $d(b_n, b) < \epsilon$. Then $a_n b_n \in U_n$ and $d(a_n b_n, ab) \leq d(a_n b_n, a_n b) + d(a_n b, ab) < 2\epsilon$. Secondly, for each $c \in G$ the conjugation map $A \rightarrow A^c$ is an isometry of $\mathcal{K}(G)$. If all the U_n are normal in G , then $U^c = (\lim_n U_n)^c = \lim_n U_n^c = \lim_n U_n = U$. Hence U is normal.

(One can avoid the hypothesis that G is compact, as long as there is a bi-invariant metric compatible with the topology: the closed normal subgroups of a Polish group G form a Polish space, being a closed subset of the Effros space $\mathcal{F}(G)$ of non-empty closed sets in G . While this space is usually seen as a Borel structure, it can be topologized using the Wijsman topology, the weakest topology that makes all the maps $C \rightarrow d(g, C)$, $C \in \mathcal{F}(G)$, continuous.)

We conclude that $\mathcal{N}(\widehat{F}_k)$ can be seen as a compact Polish space, with a compatible (complete) metric derived from a bi-invariant metric on \widehat{F}_k . We will refer to $\mathcal{N}(\widehat{F}_k)$ as the *space of k -generated profinite groups*.

For a profinite group G , by $\text{Aut}(G)$ we denote the group of continuous automorphisms. (Note that by compactness, the inverse is continuous automorphism is continuous as well. Also, if G is finitely generated as a profinite group, then every automorphism is continuous by the result of [12] mentioned above.) As described in [16, Ex. 6 on page 52], for any finitely generated profinite group G , the group $Q = \text{Aut}(G)$ is compact, and in fact profinite. To see this, let V_n be the intersection of the finitely many open subgroups of G of index at most n . Note that V_n is open and invariant in G , and the V_n form a basis of nbhds for 1_G . Since the sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ is a base of nbhds of 1, we can use it to define a compatible bi-invariant ultrametric on G : $\delta(g, h) = \inf\{2^{-n} : gh^{-1} \in V_n\}$. Note that $\delta_H(R, S) = \inf\{2^{-n} : RV_n = SV_n\}$.

Now let $W_n \trianglelefteq Q$ be the normal subgroup consisting of the $\theta \in Q$ that induce the identity on G/V_n . Clearly $\bigcap_n W_n = \{1\}$. Taking $\langle W_n \rangle_{n \in \mathbb{N}}$ as a fundamental system of nbhds of 1 in Q yields the desired topology on Q .

Lemma 2.3. *Let G be a finitely generated profinite group G , and let $Q = \text{Aut}(G)$.*

(i) *The natural action of Q on G is continuous.*

(ii) *The natural action of Q on $\mathcal{N}(G)$ given by $\theta \cdot R = \theta(R)$ is continuous.*

Proof. (i) Since Q is a Polish group and G a Polish space, it suffices to show that the action is separately continuous, namely, when we fix one argument the resulting unary map is continuous [8, I.9.14]. Firstly, each $\theta \in Q$ is (uniformly) continuous as a map $G \rightarrow G$. Secondly, for each $g \in G$, if $\theta\chi^{-1} \in W_n$, then $\delta(\theta g, \chi g) < 2^{-n+1}$, so for any $g \in G$ the map $Q \rightarrow G$ given by $\theta \rightarrow \theta g$ is continuous.

(ii) We verify separate continuity. Firstly, any $\theta \in Q$ can be extended to a continuous map $\mathcal{K}(G) \rightarrow \mathcal{K}(G)$, and hence its restriction is a continuous map $\mathcal{N}(G) \rightarrow \mathcal{N}(G)$. Secondly, for each $R \in \mathcal{N}(G)$, if $\theta\chi^{-1} \in W_n$ then by definition of the Hausdorff distance we have $\delta_H(\theta(R), \chi(R)) < 2^{-n+1}$. \square

Observe that $\text{Aut}(\widehat{F}_\omega)$ is not compact as S_∞ embeds into it as a closed subgroup. However, $\text{Aut}(\widehat{F}_\omega)$ is non-archimedean: the open subgroups W_n defined above form a nbhd base of $\{1\}$. So it is isomorphic to a closed subgroup of S_∞ .

3. COMPLEXITY OF ISOMORPHISM BETWEEN FINITELY GENERATED PROFINITE GROUPS

We view the disjoint union $\bigsqcup_{k < \omega} \mathcal{N}(\widehat{F}_k)$ as the space of finitely generated profinite groups. Note that this space becomes a Polish space by declaring a set U open if $U \cap \mathcal{N}(\widehat{F}_k)$ is open for each $k < \omega$.

Theorem 3.1. *The isomorphism relation $E_{f.g.}$ between finitely generated profinite groups is Borel equivalent to $\text{id}_{\mathbb{R}}$, the identity equivalence relation on \mathbb{R} .*

Proof. We begin by showing that $E_{f.g.} \leq_B \text{id}_{\mathbb{R}}$; this property of an equivalence relation on a Polish space is called smoothness (e.g. [5, Section 5.4]) We thank Alex Lubotzky for pointing out the crucial fact in [9, Prop. 2.2] used below to show this smoothness of $E_{f.g.}$.

Firstly let us consider the case of a fixed finite number k of generators. Write $G = \widehat{F}_k$ and as before let $Q = \text{Aut}(G)$.

For $S, T \in \mathcal{N}(\widehat{F}_k)$, we have

$$\widehat{F}_k/S \cong \widehat{F}_k/T \leftrightarrow \exists \theta \in Q [\theta(S) = T]$$

by [9, Prop. 2.2] (This depends on a Lemma of Gaschütz on lifting generating sets of finite groups. See e.g. [3, 17.7.2]). The natural action of Q on $\mathcal{N}(\widehat{F}_k)$ is continuous by Lemma 2.3. Then, since Q is compact, the orbit equivalence relation of this action on $\mathcal{N}(\widehat{F}_k)$ is closed, and hence smooth; see e.g. [5, 5.4.7].

In the general case, two given profinite groups may have a different number of generators, say $r < k < \omega$.

Lemma 3.2. *There is a continuous embedding $\phi: \mathcal{N}(\widehat{F}_r) \rightarrow \mathcal{N}(\widehat{F}_k)$ such that $\widehat{F}_r/U \cong \widehat{F}_k/\phi(U)$ for each $U \in \mathcal{N}(\widehat{F}_r)$.*

Assuming this, by the case of k generators and taking a preimage under ϕ , the relation $\{(U, S): \widehat{F}_r/U \cong \widehat{F}_k/S\}$ is closed as a subset of $\mathcal{N}(\widehat{F}_r) \times \mathcal{N}(\widehat{F}_k)$. This shows that $E_{f.g.}$ is closed on $\bigsqcup_{k < \omega} \mathcal{N}(\widehat{F}_k)$.

We verify the lemma. The embedding is as expected from the case of abstract finitely generated groups. Let P be the closed normal subgroup of \widehat{F}_k generated by $\{x_r, \dots, x_{k-1}\}$. Then $\widehat{F}_k/P = \widehat{F}_r$ and hence $\widehat{F}_k = \widehat{F}_r P$.

For $U \in \mathcal{N}(\widehat{F}_r)$ let $\langle U \rangle$ denote the closed normal subgroup of \widehat{F}_k generated by U . Note that $\langle U \rangle$ is the closure of the set of finite products $\prod_{i=1}^n u_i^{p_i}$ for $u_i \in U$ and $p_i \in P$.

Define $\phi(U) = \langle U \rangle P$. Note that $\phi(U) \in \mathcal{N}(\widehat{F}_k)$ because the product of two closed normal subgroups is closed again. As in Subsection 2.2 above, let V_n be the intersection of the finitely many open subgroups of \widehat{F}_r of index at most n , and let δ be the corresponding distance. To show ϕ is continuous, given $L \triangleleft_o \widehat{F}_k$ let n be so large that $V_n \leq L \cap \widehat{F}_r$. Suppose $U, V \in \mathcal{N}(\widehat{F}_r)$ and $\delta_H(U, V) \leq 2^{-n}$ where δ_H denotes the Hausdorff distance in \widehat{F}_r . Then $U(L \cap \widehat{F}_r) = V(L \cap \widehat{F}_r)$. We show that $\phi(U)L = \phi(V)L$. By symmetry, it suffices to verify that $u^p \in \phi(V)L$ for each $u \in U$ and $p \in P$. Now $u = v\ell$ for some $v \in V$ and $\ell \in L$. Then $u^p = v^p \ell^p \in \langle V \rangle P L = \phi(V)L$. This shows that ϕ is uniformly continuous and completes the lemma.

We prove next the converse relation $E_{f.g.} \geq_B \text{id}_{\mathbb{R}}$. A profinite group H is called *finitely presented* if $H = \widehat{F}_k/R$, $k < \omega$, and R is finitely generated as a closed normal subgroup of \widehat{F}_k . To show that $\text{id}_{\mathbb{R}} \leq_B E_{f.g.}$, we use an argument of Lubotzky [10, Prop 6.1] who showed that there are continuum many non-isomorphic profinite groups that are finitely presented as profinite groups. For a set P of primes let

$$G_P = \prod_{p \in P} \text{SL}_2(\mathbb{Z}_p) = \text{SL}_2(\widehat{\mathbb{Z}}) / \prod_{q \notin P} \text{SL}_2(\mathbb{Z}_q).$$

Here \mathbb{Z}_p is the profinite ring of p -adic integers, and $\widehat{\mathbb{Z}}$ is the completion of \mathbb{Z} , which is isomorphic to $\prod_{p \text{ prime}} \mathbb{Z}_p$. The second equality shows that G_P is finitely presented as a profinite group. Clearly the map $P \rightarrow G_P$ is Borel, and $P = Q \leftrightarrow G_P \cong G_Q$. \square

We note that Silver's dichotomy theorem, e.g. [5, 5.3.5], implies that any equivalence relation that is strictly Borel below $\text{id}_{\mathbb{R}}$ has countably many classes. So the plain result of Lubotzky [10, Prop 6.1] now already yields the Borel equivalence $E_{f.p.} \equiv_B E_{f.g.} \equiv_B \text{id}_{\mathbb{R}}$. However, by the proof of the result explained above, we in fact don't need Silver's result.

We thank Matthias Aschenbrenner for pointing out a connection to a problem due to Grothendieck. A *Grothendieck pair* is an embedding $u: G \rightarrow H$ of non-isomorphic f.p. residually finite groups such that the profinite completion $\widehat{u}: \widehat{G} \rightarrow \widehat{H}$ is an isomorphism. Bridson and Grunewald [2] showed that such pairs exist, thereby answering Grothendieck's question. We can deduce from our result a weaker form of this, which still conveys the idea that the process of profinite completion loses information.

Corollary 3.3. *There are non-isomorphic residually finite f.g. groups G, H with isomorphic profinite completions.*

Proof. A Borel equivalence relation E with all classes countable is called *weakly universal* if for each such F , there is a Borel function g such that $xFy \rightarrow g(x)Eg(y)$, and for each z the preimage $g^{-1}([z]_E)$ contains at most countably many F -classes. One can slightly modify the usual proof, based on the 0-1-law in measure theory, that almost equality of infinite bit sequences is not smooth, in order to show that no weakly universal equivalence relation is smooth.

Jay Williams [15] has proved that the isomorphism relation for f.g. groups of solvability class 3 is weakly universal. Solvable f.g. groups are residually finite (though not necessarily finitely presented). Since the process of profinite completion is Borel, the corollary now follows from Theorem 3.1. \square

4. TOWARDS COMPLEXITY OF ISOMORPHISM FOR PROFINITE GROUPS

We now consider profinite groups that aren't necessarily finitely generated. As before, we think of such a group as being given by a presentation \widehat{F}_k/N , N closed, where now $k = \omega$. Note that one has to explicitly require that isomorphisms are continuous (while continuity holds automatically for algebraic homomorphisms between f.g. profinite groups).

For a profinite group G , the commutator subgroup G' is the least closed normal subgroup S such that G/S is abelian. The closed normal subgroups $N \in \mathcal{N}(\widehat{F}_\omega)$ such that $(\widehat{F}_\omega)' \subseteq N$ form a closed subset of $\mathcal{N}(\widehat{F}_\omega)$. To see this, note that the usual algebraic commutator subgroup F'_ω is dense in it suffices to require that the usual algebraic commutator subgroup F'_ω is dense in $(\widehat{F}_\omega)'$. So it suffices to require that F'_ω is contained in N . In this way one obtains the space $\mathcal{N}_{ab}(\widehat{F}_\omega)$ of presentations of abelian profinite groups.

As pointed out by A. Melnikov, even isomorphism of these groups is quite complex. Pontryagin duality (see e.g. [7]) is a contravariant functor on the category of abelian locally compact groups G , that associates to G the group G^* of continuous homomorphisms from G into the unit circle \mathbb{T} , with the compact-open topology (which coincides with the topology inherited from the product topology if G is discrete). For a morphism $\alpha: G \rightarrow H$ let $\alpha^*: H^* \rightarrow G^*$ be the morphism defined by $\alpha^*(\psi) = \alpha \circ \psi$.

The Pontryagin duality theorem says that for each G we have $G \cong (G^*)^*$ via the map that sends $g \in G$ to the map $\theta \rightarrow \theta(g)$. A special case of this states that (discrete) abelian torsion groups A correspond to abelian profinite groups (see [13, Thm. 2.9.6] for a self-contained proof of this case). Then, as A ranges over the abelian countable torsion groups, A^* ranges over the abelian profinite groups. We have $A \cong B$ iff $A^* \cong B^*$. The duality functor and its inverse are Borel with these restrictions on the domain and range. Therefore the isomorphism relation between abelian countable torsion groups is Borel equivalent to continuous isomorphism between abelian profinite groups.

By Friedman and Stanley [4], the isomorphism relation between abelian countable torsion groups is strictly in between E_0 (a.e. equality of infinite bit sequences, which is Borel equivalent to isomorphism of rank 1 abelian

groups) and graph isomorphism (an S_∞ complete orbit equivalence relation). It is closely related to the equivalence relation $\text{id}(2^{<\omega_1})$ discussed in [5, Section 9.2], which is modelled on the classification of countable abelian torsion groups via Ulm invariants. Also see the diagram [5, p. 351] which shows that $\text{id}(2^{<\omega_1})$ is strictly between E_0 and graph isomorphism.

5. DESCRIBING PROFINITE GROUPS BY FILTERS ON A COUNTABLE LATTICE

In this section we introduce another way of representing profinite groups as points in a Polish space. We view them as filters on the lattice $\mathbb{P} = (P, \leq, \cap, \cdot)$ of open normal subgroups of \widehat{F}_ω , where

$$P = \{L \leq_o \widehat{F}_\omega : x_i \in L \text{ for almost every } i\}.$$

This is closer to the view of profinite groups as inverse limits of finite groups. Besides an application in Section 6, this view is also more appropriate for an analysis of the complexity of the isomorphism relation between profinite groups using the tools of computability theory.

When defining the lattice \mathbb{P} we can equivalently replace \widehat{F}_ω by the free group F_ω with the topology given after Def. 2.2. For, by [13, Prop 3.2.2] we have $L = \overline{(F_\omega \cap L)}$ for each $L \leq_o \widehat{F}_\omega$. Also, for each $N \leq_o F_\omega$ containing almost all the generators, we have $N = \overline{N} \cap F_\omega$.

We can effectively list without repetitions all the epimorphisms $\phi: F_\omega \rightarrow B$ where B is a finite group such that $\phi(x_i) = 1$ for almost every i . Each $L \leq_o F_\omega$ has the form $\ker \phi$, and hence can be described effectively by a single natural number in such a way that the ordering and lattice operations are computable. We denote by L_i the element of P described by i . So we can also view \mathbb{P} as a computable lattice defined on ω . Without loss of generality we may assume that $L_0 > L_2 > L_4 > \dots$ is a base of nbhds of $1 \in F_\omega$ for the topology on F_ω we given after Definition 2.2. As usual we obtain a compatible ultrametric δ on F_ω from this sequence:

$$\delta(g, h) = \inf\{2^{-n} : gh^{-1} \in L_{2n}\}.$$

Let $\mathcal{Q}(\mathbb{P})$ denote the set of filters of \mathbb{P} . By the remark above we can view $\mathcal{Q}(\mathbb{P})$ as an effectively closed (that is, Π_1^0) set in Cantor space $2^\mathbb{N}$, and hence as a Polish space. In the following the variables R, S range over elements of $\mathcal{N}(\widehat{F}_\omega)$, and L over elements of P . For $Z \in 2^\mathbb{N}$ by $Z \upharpoonright_n$ we denote the string consisting of the first n bits of Z . If σ is a string of length n , by $[\sigma]$ we denote the set $\{Z \in 2^\mathbb{N} : Z \upharpoonright_n = \sigma\}$.

Lemma 5.1.

$$\delta_H(R, S) \leq 2^{-n} \Leftrightarrow RL_{2n} = SL_{2n} \Leftrightarrow \forall L \geq L_{2n} [R \leq L \leftrightarrow S \leq L].$$

Proof. For the first equivalence, clearly, for each $u \in R$ there is $v \in S$ such that $uv^{-1} \in L_{2n}$ iff $R \leq SL_{2n}$. Symmetrically, for each $v \in S$ there is $u \in R$ such that $uv^{-1} \in L_{2n}$ iff $R \leq SL_{2n}$.

The second equivalence is clear because RL_{2n} is the least $L \geq R, L_{2n}$. \square

Lemma 5.2. *There is a continuous isomorphism of lattices $\Phi: \mathcal{N}(\widehat{F}_\omega) \rightarrow \mathcal{Q}(\mathbb{P})$ given by*

$$\begin{aligned}\Phi(R) &= \{L: N \leq L\}. \text{ Its inverse is} \\ \Theta(\mathcal{G}) &= \bigcap \mathcal{G}.\end{aligned}$$

Proof. Clearly Φ and Θ are inverses. To show that Φ is (uniformly) continuous, given $s \in \omega$, let n be least so that $L_{2n} \leq L_i$ for each $i < s$. We have $\delta_H(R, S) \leq 2^{-n} \Leftrightarrow RL_{2n} = SL_{2n} \Rightarrow \Phi(R) \in [\sigma] \Leftrightarrow \Phi(S) \in [\sigma]$ for each string σ of length s .

This already implies that Θ is (uniformly) continuous by compactness of the two spaces. However, we can also obtain explicit bounds for the uniform continuity: given n , suppose $Z, W \in \mathcal{Q}(\mathbb{P})$ and $Z \upharpoonright_{r+1} = W \upharpoonright_{r+1}$ where $r = \max\{i: L_i \geq L_{2n}\}$. Then $\Theta(Z)L_{2n} = \Theta(W)L_{2n}$ and hence $\delta(\Theta(Z), \Theta(W)) \leq 2^{-n}$. \square

6. AN UPPER BOUND ON THE COMPLEXITY OF ISOMORPHISM BETWEEN PROFINITE GROUPS

Theorem 6.1. *Isomorphism of profinite groups is classifiable by countable structures.*

Proof. Let \cong denote isomorphism of profinite groups, viewed as a relation on $\mathcal{N}(\widehat{F}_\omega)$. We will determine a closed subgroup V of S_∞ and a continuous V -action on a Polish space X such that $(\mathcal{N}(\widehat{F}_\omega), \cong)$ is Borel below the orbit equivalence relation E_V^X . This will suffice for the theorem by a folklore fact from descriptive set theory provided in Lemma 6.2 below.

To obtain V and its action on a Polish space X , we use that every infinite profinite group $G = \widehat{F}_\omega/R$ is homeomorphic to Cantor space $2^\mathbb{N}$. So we can view the group operations as continuous functions on Cantor space. X is the space of continuous binary functions on Cantor space that encode a “group difference operation” $\rho(a, b) = ab^{-1}$ where 0^ω is the identity. V is the group of homeomorphisms of Cantor space that fix 0^ω . V acts continuously on X . The group V can be seen naturally as a closed subgroup of S_∞ via the usual Stone duality. Given an infinite profinite group $G = \widehat{F}_\omega/R$, the task is to obtain a homeomorphism of G with Cantor space sending the identity to 0^ω . This yields a homeomorphic copy ρ_R of the group difference operation as a continuous binary function on Cantor space. Once we can do this, we have $\widehat{F}_\omega/R \cong \widehat{F}_\omega/S \Leftrightarrow$ there is $g \in V$ sending ρ_R to ρ_S , as required.

We describe how to accomplish this task. Via Lemma 5.2 we may view $R \in \mathcal{N}(\widehat{F}_\omega)$ as a filter on \mathbb{P} . The principal filters form a countable (F_σ) set in $\mathcal{Q}(\mathbb{P})$, which we can ignore in the Borel reduction. So we may further assume that R is non-principal. Using R as a Turing oracle we may effectively obtain a sequence $F_\omega = L_{i_0} > L_{i_1} > \dots$ generating R as a filter. In other words, $\bigcap_n L_{i_n} = R$ when R is viewed as an element of $\mathcal{N}(\widehat{F}_\omega)$.

We write $S_n = L_{i_n}$. To obtain the homeomorphism of \widehat{F}_ω/R as a topological space with Cantor space, for each n we can effectively determine $k_n = |S_n : S_{n+1}|$ and a sequence $\langle g_i^{(n)} \rangle_{i < k_n}$ of coset representatives for S_{n+1} in S_n such that $g_0^{(n)} = 1$.

Let T be the tree of strings $\sigma \in \omega^{<\omega}$ such that $\sigma(i) < k_i$ for each $i < |\sigma|$. For $|\sigma| = n$ we have a coset

$$(1) \quad C_\sigma = g_{\sigma(0)}^{(0)} g_{\sigma(1)}^{(1)} \cdots g_{\sigma(n-1)}^{(n-1)} S_n.$$

The clopen sets C_σ form a basis for \widehat{F}_ω/R . In this way \widehat{F}_ω/R is naturally homeomorphic to $[T]$ where the identity element corresponds to 0^ω .

For $Z \in [T]$, by $Z \upharpoonright_n$ we denote the string consisting of the first n entries of Z . The group difference operation $a, b \rightarrow ab^{-1}$ of $G = \widehat{F}_\omega/R$ can now be seen as a continuous map $[T] \times [T] \rightarrow [T]$ given by $(Z, W) \mapsto Y$ where, via the identifications of strings of length n with their coset representatives for F_ω/S_n given by (1), we have

$$(Z \upharpoonright_n S_n)(W \upharpoonright_n S_n) = Y \upharpoonright_n S_n \text{ for each } n.$$

The standard homeomorphism $[T] \rightarrow 2^\mathbb{N}$ is defined by $Z \rightarrow \bigcup_n F(Z \upharpoonright_n)$, where F is the computable map defined as follows. Let $F(\langle \rangle) = \langle \rangle$. If $F(\sigma)$ has been defined where $|\sigma| = n$, let $F(\sigma 0) = F(\sigma)0^{k_n}$ and $F(\sigma i) = F(\sigma)0^{k_n-i}1$ for $0 < i < k_n$. Note that 0^ω is mapped to 0^ω . Via this homeomorphism the group difference operation is turned into a continuous map $\rho_R: 2^\mathbb{N} \times 2^\mathbb{N} \rightarrow 2^\mathbb{N}$.

Cantor space $2^\mathbb{N}$ is equipped with the usual ultrametric d . The space of continuous functions $\mathcal{C}(2^\mathbb{N} \times 2^\mathbb{N}, 2^\mathbb{N})$ is a Polish space via the supremum distance based on d . Let X be the closed subspace of this space consisting of the binary operations ρ that satisfy the group axioms, written in terms of the group difference operation, with 0^ω as the identity element. (That is, substitute $\rho(0^\omega, a)$ for the inverse operation, and $\rho(a, \rho(0^\omega, b))$ for the group operation.) The desired Polish group V is the stabiliser of 0^ω in the Polish group $\mathcal{H}(2^\mathbb{N})$, the homeomorphisms of Cantor space with the topology inherited from $\mathcal{C}(2^\mathbb{N}, 2^\mathbb{N})$, which canonically acts on X via

$$(g \cdot \rho)(u, v) = g\rho(g^{-1}u, g^{-1}v).$$

Clearly, for $R, S \in \mathcal{N}(\widehat{F}_\omega)$ of infinite index in \widehat{F}_ω , we have

$$\widehat{F}_\omega/R \cong \widehat{F}_\omega/S \leftrightarrow \exists g \in V [g \cdot \rho_R = \rho_S].$$

A homeomorphism of Cantor space is given by its action on the dense countable Boolean algebra D of clopen sets. So V is continuously isomorphic to a closed subgroup of $\mathbf{Aut}(D)$, and hence of S_∞ , as required. \square

Lemma 6.2. *Let V be a closed subgroup of S_∞ with a Borel action on a Polish space X . Then E_V^X is classifiable by countable structures.*

Proof. The proof is in two steps. The first step adapts the proof of Becker and Kechris [1, 2.7.4] in the version of Gao [5, 3.6.1].

For a Polish group G and Borel actions of G on Polish space X, Y , a *Borel G -embedding* is a Borel map $\theta: X \rightarrow Y$ such that $g \cdot \theta(x) = \theta(g \cdot x)$ for each $x \in X$. Let $\mathcal{F}(G)$ be the usual Effros space consisting of the closed subsets of G . By [1, 2.6.1] $\mathcal{F}(G)^\omega$, with the canonical Borel G -action, is universal for Borel G -actions under Borel G -embeddings.

For a countable signature \mathcal{L} , let $\mathbf{Mod}(\mathcal{L})$ denote the Polish space of \mathcal{L} -structures with domain ω . Note that S_∞ acts continuously on $\mathbf{Mod}(\mathcal{L})$. In

Gao [5, 3.6.1] an (infinitary) signature \mathcal{L}_1 is provided, together with a Borel S_∞ -embedding of $\mathcal{F}(S_\infty)^\omega$ into $\text{Mod}(\mathcal{L}_1)$. Restricting this yields a Borel V -embedding θ of $\mathcal{F}(V)^\omega$ into $\text{Mod}(\mathcal{L}_1)$.

For the second step of the proof we use the construction in [1, Section 1.5] (also see the discussion around [1, 2.7.4]). For a signature \mathcal{L}_2 , which we may assume to be disjoint from \mathcal{L}_1 , this yields an \mathcal{L}_2 -structure M with domain ω such that $\text{Aut}(M) = V$. Given $x \in X = \mathcal{F}(V)^\omega$, let N_x be the $\mathcal{L}_1 \cup \mathcal{L}_2$ -structure with domain ω which has all the relations and functions of $\theta(x)$ and M . If $\rho \in S_\infty$ shows that $N_x \cong N_y$ then ρ is an automorphism of M , and hence $\rho \in V$. Therefore $xE_V^X y \leftrightarrow N_x \cong N_y$.

The lemma follows because the Borel action of V on $\mathcal{F}(V)^\omega$ is universal for V -embeddings. \square

7. ISOMORPHISM BETWEEN PROFINITE GROUPS IS BOREL COMPLETE FOR S_∞ ORBIT EQUIVALENCE RELATIONS

Recall that a group G is nilpotent of class 2 (nil-2 for short) if it satisfies the law $[[x, y], z] = 1$. Equivalently, the commutator subgroup is contained in the center. For a prime p , the group of unitriangular matrices

$$\text{UT}_3^3(\mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}$$

is an example of a nil-2 group of exponent p .

Theorem 7.1. *Let $p \geq 3$ be a prime. Any S_∞ orbit equivalence relation can be Borel reduced to isomorphism between profinite nil-2 groups of exponent p .*

Proof. The main result in Mekler [11] implies a version of Thm. 7.1 for abstract, rather than profinite, groups. Mekler associates to each symmetric and irreflexive graph A a nil-2 exponent- p group $G(A)$ in such a way that isomorphic graphs yield isomorphic groups. In the countable case, the map G sends a countable graph A to a countable group $G(A)$ in a Borel way.

Definition 7.2. A symmetric and irreflexive graph is called *nice* if it has no triangles, no squares, and for each pair of distinct vertices x, y , there is a vertex z joined to x and not to y .

Mekler [11] proves that a nice graph A can be interpreted in $G(A)$ using first-order formulas without parameters (see [6, Ch. 5] for background on interpretations). In particular, for nice graphs A, B we have $A \cong B$ iff $G(A) \cong G(B)$. (For a more detailed write-up see [6, A.3].) Since isomorphism of nice graphs is S_∞ -complete, so is isomorphism of countable nil-2 exponent p groups.

Our proof is based on Mekler's, replacing the groups $G(A)$ he defined by their completions $\widehat{G}(A)$ with respect to a suitable basis of nbhds of the identity.

In the following all graphs will be symmetric and irreflexive, and have domain ω . Such a graph is thus given by its set of edges $A \subseteq \{\langle r, s \rangle : r < s\}$. We write rAs (or simply rs if A is understood) for $\langle r, s \rangle \in A$.

Let F be the free nil-2 exponent- p group on free generators x_0, x_1, \dots . For $r \neq s$ we write

$$x_{r,s} = [x_r, x_s].$$

As noted in [11], the centre $Z(F)$ of F is an abelian group of exponent p that is freely generated by the $x_{r,s}$ for $r < s$. Given a graph A , Mekler lets

$$G(A) = F / \langle x_{r,s} : rAs \rangle.$$

In particular $F = G(\emptyset)$. The centre $Z = Z(G(A))$ is an abelian group of exponent p freely generated by the $x_{r,s}$ such that $\neg rAs$. Also $G(A)/Z$ is an abelian group of exponent p freely generated by the Zx_i . (Intuitively, when defining $G(A)$ as a quotient of F , the commutators $x_{r,s}$ such that rAs vanish, but no vertices vanish.)

Lemma 7.3 (Normal form for $G(A)$, [11, 6]). *Every element c of Z can be written uniquely in the form $\prod_{\langle r,s \rangle \in L} x_{r,s}^{\beta_{r,s}}$ where $L \subseteq \omega \times \omega$ is a finite set of pairs $\langle r, s \rangle$ with $r < s$ and $\neg rAs$, and $0 < \beta_{r,s} < p$.*

Every element of $G(A)$ can be written uniquely in the form $c \cdot v$ where $c \in Z$, and $v = \prod_{i \in D} x_i^{\alpha_i}$, for $D \subseteq \omega$ finite and $0 < \alpha_i < p$. (The product $\prod_{i \in D} x_i^{\alpha_i}$ is interpreted along the indices in ascending order.)

Given a graph A , let R_n be the normal subgroup of $G(A)$ generated by the x_i , $i \geq n$. Note that F/R_n is finite, being a f.g. nilpotent torsion group. Let $\widehat{G}(A)$ be the completion of $G(A)$ with respect to the set $\mathcal{V} = \{R_n : n \in \omega\}$ (see Subsection 2.1). By Lemma 7.3 we have $\bigcap_n R_n = \{1\}$, so $G(A)$ embeds into $\widehat{G}(A)$.

In set theory one inductively defines $0 = \emptyset$ and $n = \{0, \dots, n-1\}$ to obtain the natural numbers; this will save on notation here. A set of coset representatives for $G(A)/R_n$ is given by the $c \cdot v$ as in Lemma 7.3, where $D \subseteq n$ and $E \subseteq n \times n$. The completion $\widehat{G}(A)$ of $G(A)$ with respect to the R_n consists of the maps $\rho \in \prod_n G(A)/R_n$ such that $\rho(gR_{n+1}) = gR_n$ for each $n \in \omega$ and $g \in G(A)$.

If $\rho(gR_{n+1}) = hR_n$ where $h = c \cdot v$ is a coset representative for R_n , then we can define a coset representative $c' \cdot v'$ for gR_{n+1} as follows: we obtain c' from c by potentially appending to c factors involving the $x_{r,n}$ for $r < n$, and v' from v by potentially appending a factor $x_n^{\alpha_n}$. So we can view ρ as given by multiplying two formal infinite products:

Lemma 7.4 (Normal form for $\widehat{G}(A)$). *Every $c \in Z(\widehat{G}(A))$ can be written uniquely in the form $\prod_{\langle r,s \rangle \in L} x_{r,s}^{\beta_{r,s}}$ where $L \subseteq \omega \times \omega$ is a set of pairs $\langle r, s \rangle$ with $r < s$, $\neg rAs$, and $0 < \beta_{r,s} < p$.*

Every element of $\widehat{G}(A)$ can be written uniquely in the form $c \cdot v$, where $v = \prod_{i \in D} x_i^{\alpha_i}$, $c \in Z(\widehat{G}(A))$, $D \subseteq \omega$, and $0 < \alpha_i < p$ (the product is taken along ascending indices).

We can define the infinite products above explicitly as limits in $\widehat{G}(A)$. We view $G(A)$ as embedded into $\widehat{G}(A)$. Given formal products as above, let

$$v_n = \prod_{i \in D \cap n} x_i^{\alpha_i} \text{ and } c_n = \prod_{\langle r,s \rangle \in L \cap n \times n} x_{r,s}^{\beta_{r,s}}.$$

For $k \geq n$ we have $v_k v_n^{-1} \in \overline{R_n}$ and $c_k c_n^{-1} \in \overline{R_n}$. So $v = \lim_n v_n$ and $c = \lim_n c_n$ exist and equal the values of the formal products as defined above.

Each nil-2 group satisfies the distributive law $[x, yz] = [x, y][x, z]$. This implies that $[x_r^\alpha, x_s^\beta] = x_{r,s}^{\alpha\beta}$. The following lemma generalises to infinite products the expression for commutators that were obtained using these identities in [11, p. 784] (and also in [6, proof of Lemma A.3.4]).

Lemma 7.5 (Commutators). *The following holds in $\widehat{G}(A)$.*

$$[\prod_{r \in D} x_r^{\alpha_r}, \prod_{s \in E} x_s^{\beta_s}] = \prod_{r \in D, s \in E, r < s, \neg rs} x_{r,s}^{\alpha_r \beta_s - \alpha_s \beta_r}$$

Proof. Based on the case of finite products, by continuity of the commutator operation and using the expressions for limits above, we have

$$\begin{aligned} [\prod_{r \in D} x_r^{\alpha_r}, \prod_{s \in E} x_s^{\beta_s}] &= [\lim_n \prod_{r \in D \cap n} x_r^{\alpha_r}, \lim_n \prod_{s \in E \cap n} x_s^{\beta_s}] \\ &= \lim_n \prod_{r \in D, s \in E, r < s < n, \neg rs} x_{r,s}^{\alpha_r \beta_s - \alpha_s \beta_r} \\ &= \prod_{r \in D, s \in E, r < s, \neg rs} x_{r,s}^{\alpha_r \beta_s - \alpha_s \beta_r}. \end{aligned}$$

□

The following is a direct consequence of Lemma 7.5. $C(g)$ denotes the centraliser of a group element g .

Lemma 7.6. *Let $v \in \widehat{G}(A)$. If $0 < \gamma < p$ we have $C(v^\gamma) = C(v)$.*

Mekler's argument employs the niceness of A to show that a copy of the set of vertices of the given graph is first-order definable in $G(A)$. The copy of vertex i is a certain definable equivalence class of the generator x_i . He provides a first-order interpretation Γ_2 without parameters such that $\Gamma_2(G(A)) \cong A$. We will show that his interpretation has the same effect in the profinite case: $\Gamma_2(\widehat{G}(A)) \cong A$.

We first summarize Mekler's interpretation Γ_2 . Let H be a group with centre $Z(H)$.

- For $a \in H$ let \bar{a} denote the coset $aZ(H)$.
- Write $\bar{a} \sim \bar{b}$ if $C(a) = C(b)$. Let $[\bar{a}]$ be the \sim equivalence class of \bar{a} . (Thus, for $c \in Z(H)$ we have $[\bar{c}] = \{Z(H)\}$.)
- Slightly deviating from Mekler's notation, let

$$[\bar{a}]R[\bar{b}] \leftrightarrow [\bar{a}], [\bar{b}], [\bar{1}] \text{ are all distinct and } [a, b] = 1.$$

Let

$$\Gamma_1(H) = \langle \{[\bar{a}] : a \in H \setminus Z(H)\}, R \rangle.$$

$$\Gamma_2(H) = \{[\bar{a}] : |[\bar{a}]| = p - 1 \wedge \exists \bar{w} [\bar{a}]R[\bar{w}]\},$$

viewed as a subgraph of $\Gamma_1(H)$.

We are ready to verify that a nice graph can be recovered from its profinite group via the interpretation Γ_2 .

Lemma 7.7. *For a nice graph A , we have $\Gamma_2(\widehat{G}(A)) \cong A$ via $[\bar{x}_i] \mapsto i$.*

We follow the outline of the proof of [11, Lemma 2.2]. The notation there will be adapted to $\widehat{G}(A)$ via allowing infinite products.

Henceforth, in products of the form $\prod_{i \in D} x_i^{\alpha_i}$ we will assume that $D \subseteq \omega$ is nonempty and $0 < \alpha_i < p$ for each $i \in D$. We also set $\alpha_i = 0$ for $i \notin D$. Expressions and calculations involving number exponents α will all be modulo p ; e.g. $\alpha \neq 0$ means that $\alpha \not\equiv 0 \pmod{p}$.

All we need to show is that the formula in Γ_2 defines the set $\{[\bar{x}_i] : i \in \omega\}$. Suppose some \bar{v} is given where $v \notin Z(\widehat{G}(A))$. We may assume that $v = \prod_{i \in D} x_i^{\alpha_i}$ satisfying the conventions above. We will show that

$$[\bar{v}] = [\bar{x}_k] \text{ for some } k \leftrightarrow |\bar{v}| = p - 1 \wedge \exists \bar{w} [\bar{v}]R[\bar{w}].$$

We distinguish four cases, and verify the equivalence in each case. In Case 1 we check that the formula holds, in Case 2-4 that it fails. Recall that rs is short for rAs , i.e. that vertices r, s are joined.

Case 1: $D = \{r\}$ for some r .

For the first condition of the formula in Γ_2 , note that we have $\bar{v} = \bar{x}_r^\alpha$ for some $\alpha \neq 0$. Suppose $\bar{u} \sim \bar{v}$ for $u = \prod_{k \in E} x_k^{\beta_k}$. Then $E = \{r\}$. For, if $k \in E$, $k \neq r$, then by niceness of A there is an s such that $rs \wedge \neg ks$, so that $x_s \in C(v) \setminus C(w)$ by Lemma 7.5. Thus $\bar{u} = \bar{x}_r^\beta$ with $0 < \beta < p$.

For the second condition, by niceness of A we pick $k \neq i$ such that ik , and let $\bar{w} = \bar{x}_k$.

Case 2: $D = \{r, s\}$ for some r, s such that rAs (and hence $r \neq s$).

We show that $[\bar{v}] = \{\bar{x}_r^\alpha \bar{x}_s^\beta : 0 < \alpha, \beta < p\}$, and hence this equivalence class has size $(p - 1)^2$. To this end we verify:

Claim. Let $w = \prod_{k \in E} x_k^{\beta_k}$. For each $\alpha, \beta \neq 0$, we have

$$[w, \bar{x}_r^\alpha \bar{x}_s^\beta] = 1 \leftrightarrow E \subseteq \{r, s\}.$$

For the implication from left to right, if there is $k \in E \setminus \{r, s\}$, then kr and ks , so A has a triangle. Hence $E \subseteq \{r, s\}$. The converse implication follows by distributivity and since $[x_r, x_s] = 1$. This shows the claim.

Case 3: Neither Case 1 nor 2, and there is an ℓ such that $i\ell$ for each $i \in D$.

Claim. $[\bar{v}] = \{\bar{v}^\gamma \bar{x}_\ell^\beta : 0 < \gamma < p \wedge 0 \leq \beta < p\}$, and hence has size $p(p - 1)$.

First an observation: suppose that $[v^\gamma x_\ell^\beta, w] = 1$ where γ, β are as above and $w = \prod_{k \in E} x_k^{\beta_k}$. Then $E \subseteq D \cup \{\ell\}$. For, since Case 1 and 2 don't apply, there are distinct $p, q \in D \setminus \{\ell\}$. Since A has no squares, ℓ is the only vertex adjacent to both p and q . Hence, given $j \in E \setminus \{\ell, p, q\}$, we have $\neg pj \vee \neg qj$, say the former. Then $j \in D$, for otherwise, in Lemma 7.5, in the expansion of $[v^\gamma x_\ell^\beta, w]$, we get a term $x_{p,j}^m$ with $m \neq 0$ and $\neg pj$ (assuming that $p < j$, say).

The inclusion “ \supseteq ” of the claim now follows by Lemma 7.5. For the inclusion “ \subseteq ” suppose that $[v, w] = 1$ where $w = \prod_{k \in E} x_k^{\beta_k}$. Then $E \subseteq D \cup \{\ell\}$ by our observation. By Lemma 7.5 we now have

$$[v, w] = \prod_{r, s \in D \setminus \{\ell\}, r < s, \neg rs} x_{r,s}^{\alpha_r \beta_s - \alpha_s \beta_r}.$$

Let $m = \min(D \setminus \{\ell\})$. Since $\alpha_m \neq 0$ we can pick γ such that $\beta_m = \gamma\alpha_m$. Since A has no triangles we have $\neg rs$ for any $r < s$ such that $r, s \in D \setminus \{\ell\}$, and hence $\alpha_r\beta_s = \alpha_s\beta_r$. By induction on the elements s of $D \setminus \{\ell\}$ this yields $\beta_s = \gamma\alpha_s$: if we have it for some $r < s$ in $D \setminus \{\ell\}$ then $\beta_r\beta_s = \gamma\alpha_r\beta_s = \gamma\alpha_s\beta_r$. Hence $\beta_s = \gamma\alpha_s$, for if $\beta_r \neq 0$ we can cancel it, and if $\beta_r = 0$ then also $\beta_s = 0$ because $\alpha_r \neq 0$.

This shows that $\bar{w} = \bar{v}^\gamma \bar{x}_\ell^\beta$ for some β . In particular, $C(v) = C(w)$ implies that \bar{w} has the required form.

Case 4: Neither Case 1, 2 or 3.

Claim. $[\bar{v}] = \{\bar{v}^\gamma : 0 < \gamma < p\}$, so this class has $p - 1$ elements.

There is no \bar{w} such that $[\bar{v}]R[\bar{w}]$.

The inclusion “ \supseteq ” of the first statement follows from Lemma 7.6. We now verify the converse inclusion and the second statement. By case hypothesis there are distinct $\ell_0, \ell_1 \in D$ such that $\neg\ell_0\ell_1$. Since A has no squares there is at most one $q \in D$ such that $\ell_0q \wedge \ell_1q$. If q exists, as we are not in Case 3 we can choose $q' \in D$ such that $\neg q'q$.

We define a linear order \prec on D , which is of type ω if D is infinite. It begins with ℓ_0, ℓ_1 , and is followed by q', q if they are defined. After that we proceed in ascending order for the remaining elements of D . Then for each $v \in D \setminus \{\min D\}$ there is $u \prec v$ in D (in fact among the first three elements) such that $\neg uv$.

Suppose now that $[v, w] = 1$ where $w = \prod_{k \in E} x_k^{\beta_k}$. Then $E \subseteq D$: if $s \in E \setminus D$ there is $r \in D$ such that $\neg rs$. This implies $\alpha_r\beta_s = \alpha_s\beta_r$, but $\alpha_s = 0$ while the left hand side is $\neq 0$, contradiction.

By a slight variant of Lemma 7.5, using that \prec eventually agrees with $<$, we now have $[v, w] = \prod_{r, s \in D, r \prec s, \neg rs} x_{r, s}^{\alpha_r\beta_s - \alpha_s\beta_r}$. Choose γ such that $\gamma\alpha_{\ell_0} = \beta_{\ell_0}$. By induction along (D, \prec) we see that $\beta_s = \gamma\alpha_s$ for each $s \in D$: if $\ell_0 \prec s$ choose $r \prec s$ such that $\neg rs$. Since $\alpha_r\beta_s = \alpha_s\beta_r$, as in Case 3 before we may conclude that $\beta_s = \gamma\alpha_s$.

This shows that $\bar{v}^\gamma = \bar{w}$. Further, if $[\bar{w}] \neq [\bar{1}]$ then $\gamma \neq 0$ so that $[\bar{v}] = [\bar{w}]$, as required. \square

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